A model of Myerson-Nash equilibria in networks

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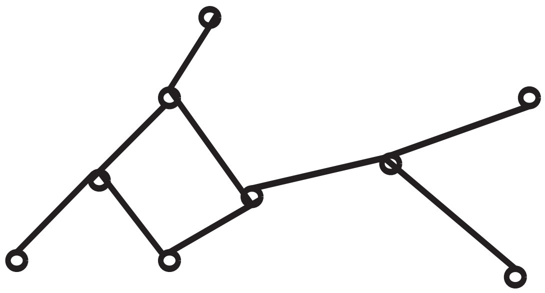
Network formation

Jackson & Wolinski (1996)

Three points to solve:

1. a surplus (*value function* V) from the network;
2. a distribution (*allocation rule* $\mathcal{A}$) of this payoff;
3. a definition of strategies and equilibrium (stability).



N agents
a link costs them k

Value function

The *connected couples* (gross) value function V^{CC} is:

$$V^{CC}(G) \equiv |\{(a,b) \in \mathcal{N} \times \mathcal{N} : a \bowtie b\}| = \sum_{C \sqsubset_G \mathcal{N}} \frac{|C|(|C|-1)}{2} \quad .$$

If G is connected $V^{CC}(G) = \frac{N(N-1)}{2}$.

Considering a fixed link formation cost of k, the (net) value function is:

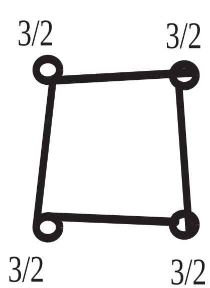
$$V^{CC}(G) - 2 \cdot k \cdot L(G) \quad .$$

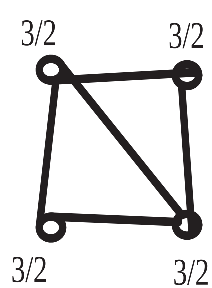
Example - $\mathcal{A}^{EN}$

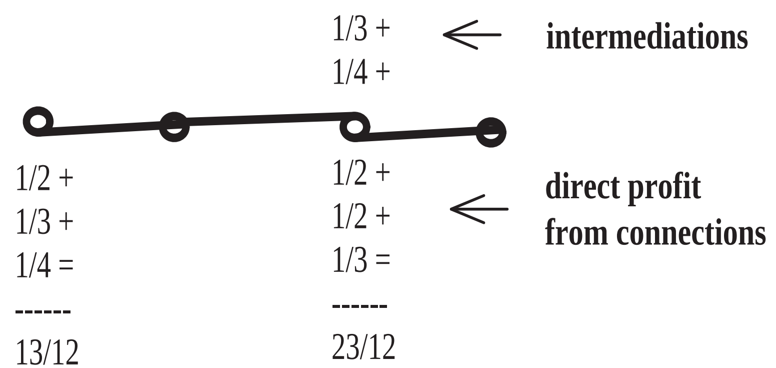
The *essential nodes* (EN) allocation rule $\mathcal{A}^{EN}$:

$$\mathcal{A}_a^{EN}(G) = \sum_{a \bowtie_G b} \frac{1}{2 + |\{c : a \overset{c}{\bowtie}_G b\}|} + \sum_{b \overset{a}{\bowtie}_G c} \frac{1}{2 + |\{d : b \overset{d}{\bowtie}_G c\}|} \quad .$$

1 unit produced by *any connected couple* is divided equally between them and all the *essential* nodes between them:







Myerson value as allocation rule

The *Myerson value* (MV) allocation rule $\mathcal{A}^{MV}$ is:

$$\mathcal{A}_a^{MV}(G,V) = \sum_{S\subseteq\mathcal{N}:\ a\in S} \frac{(|S|-1)!(N-|S|)!}{N!}\Big[V_G(S) - V_G(S\setminus\{a\})\Big]$$

- Anonimity: invariance under permutations.
- Fairness: changing the status of a link has the same effect for its two nodes.

Myerson '77: $\mathcal{A}^{MV}$ is the only *anonimous* and *fair* all. rule.

$\mathcal{A}^{MV^{CC}} - k \cdot L(a)$ is the only one for $V^{CC}(G) - 2 \cdot k \cdot L(G)$.

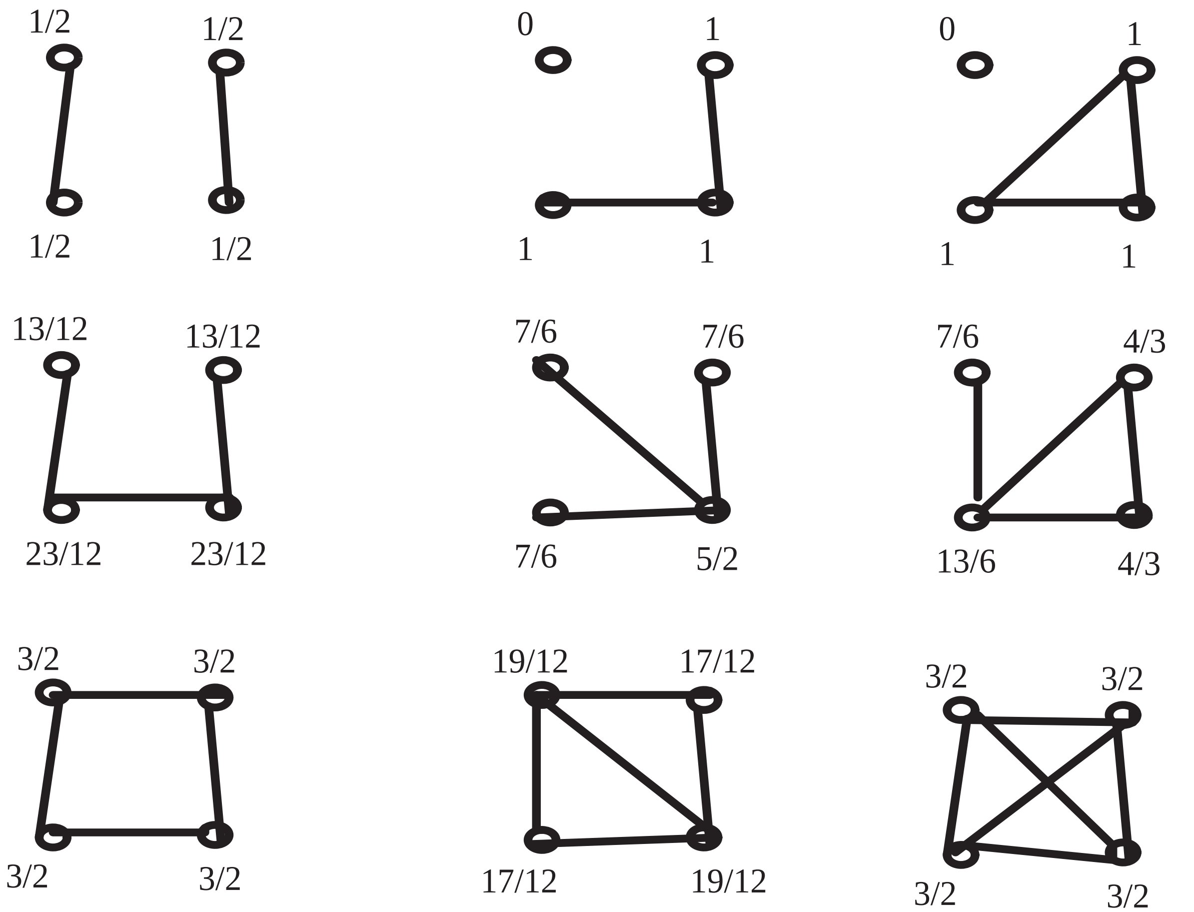
Myerson value

MV is a result of cooperative game theory, but in specific frameworks also the limit of non-cooperative bargaining:

- Gul 1989;
- Navarro & Perea 2002.

MV requires to check all clusters for all the 2^N possible subsets $\Longrightarrow$ NP-hard.

Myerson value - $N = 4$



k is not considered

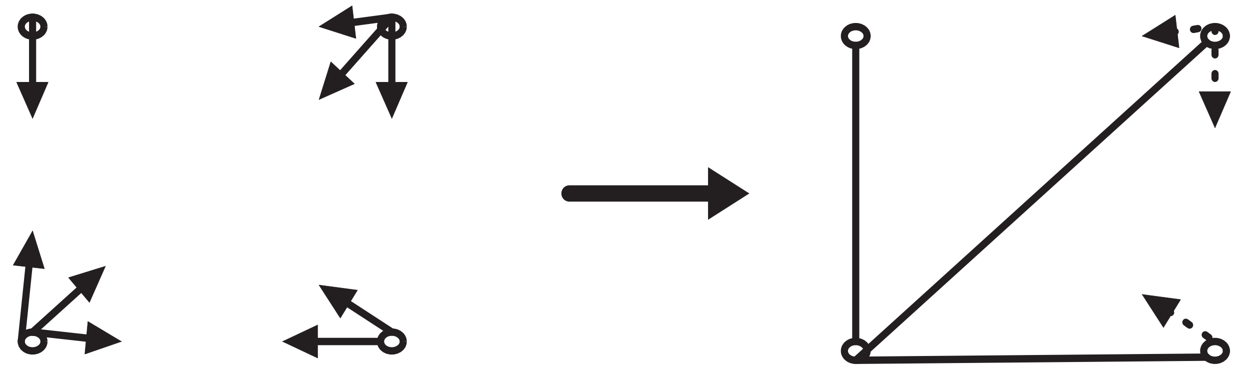
Link proposal strategies

a's *link proposal* strategy is a vector $\vec{a} \in \{0,1\}^{N-1}$

$\vec{a}_b = 1$ indicates that a is willing to form a link with b

A link $g_{a,b}$ is formed only if $\vec{a}_b = \vec{b}_a = 1$.

Example:



Possible Equilibria

Pairwise Stability [Jackson & Wolinski (1996)]:

- No agent wants to erase *one* of her links,
- no two unlinked agents want to link.

Strong Pairwise Stability [Belleflamme & Bloch (2004)]:

- No agent wants to erase *any subset* of her links,
- no two unlinked agents want to link.

$\longrightarrow$ For the second point of both we can use *fairness*.

Myerson-Nash equilibrium

Formally:

Being $\mathcal{A}_a^{MV}(G, V^{CC}) - k \cdot l(a)$ the (net) allocation rule, G is a *Myerson-Nash* equilibrium (MNE) iff:

$$\forall\, a, b \in \mathcal{N},\; b \neq a,\; \mathcal{A}_a^{MV}(G, V^{CC}) \geq \mathcal{A}_a^{MV}(G \cup \{g_{a,b}\}, V^{CC}) - k$$

$$\wedge$$

$$\forall\, a \in \mathcal{N},\; \forall\, \Gamma \subseteq \{\eta \in G :\; a \in \eta\},$$

$$\mathcal{A}_a^{MV}(G, V^{CC}) \geq \mathcal{A}_a^{MV}(G \backslash \Gamma, V^{CC}) + k \cdot |\Gamma|\;.$$

Theoretical results

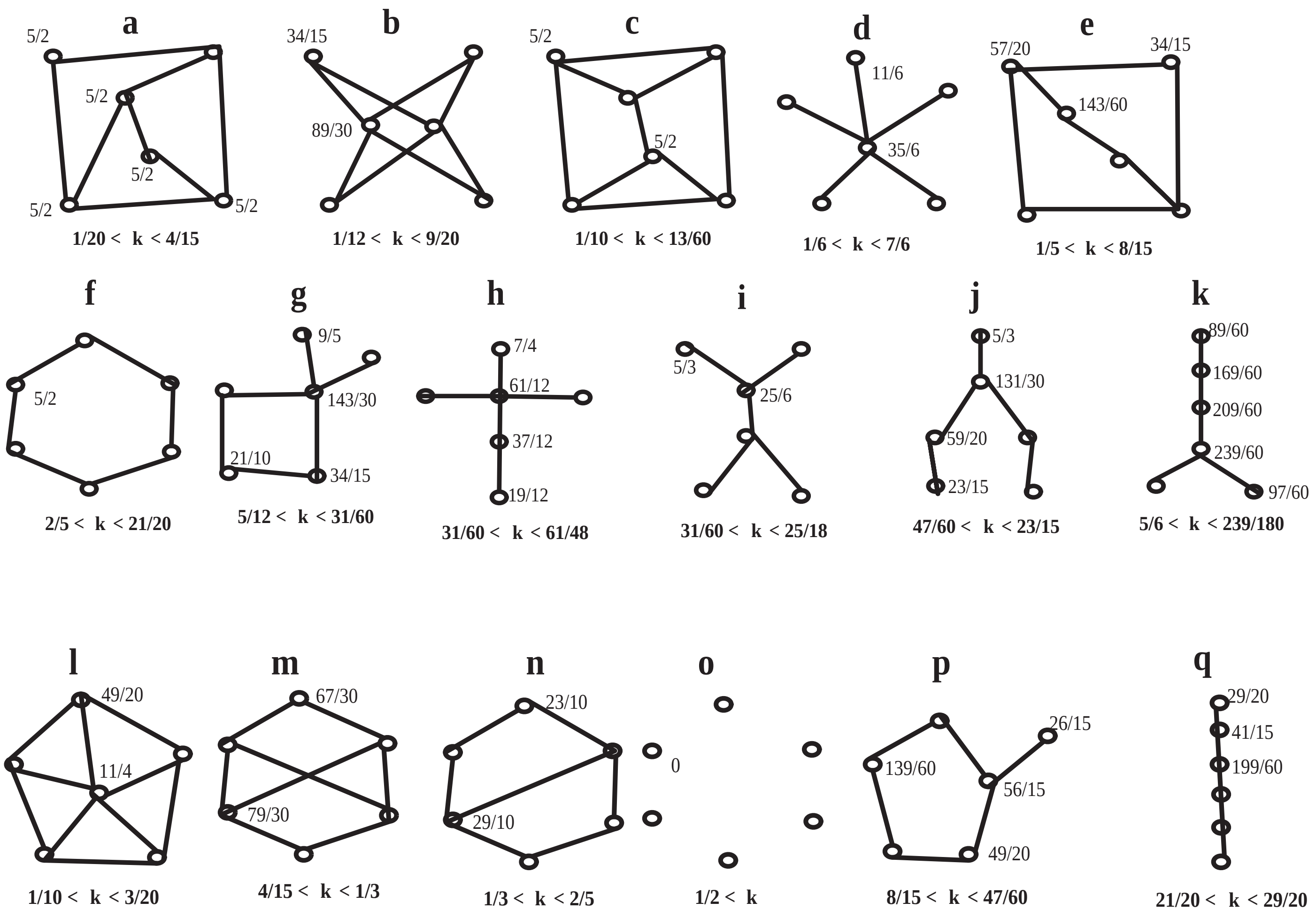
We have three main propositions:

- Every MNE is either the empty network or a connected graph.
- The empty network is MNE for $k \geq \frac{1}{2}$, the complete network is for $k \leq \frac{1}{N(N-1)}$, the star is for $\frac{1}{6} \leq k \leq \frac{N+1}{6}$.
- Under V^{CC}, in general, $\mathcal{A}^{MV} \neq \mathcal{A}^{EN}$, however, when G is a tree, the two coincide.

And a conjecture:

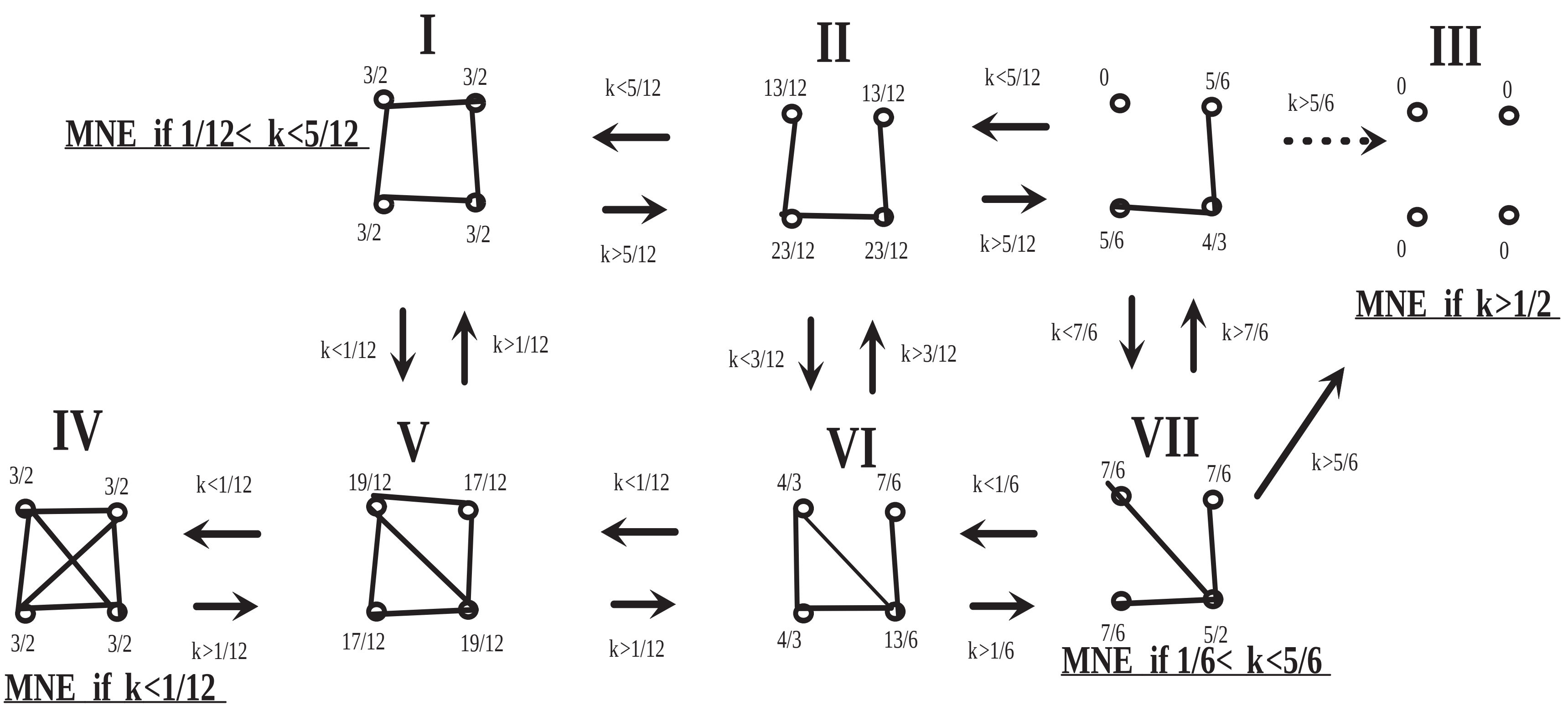
- For every MNE G, $\forall\, k$, $\forall\, N$, $D_G \leq 6$.

$N = 6$ - *some equilibria*

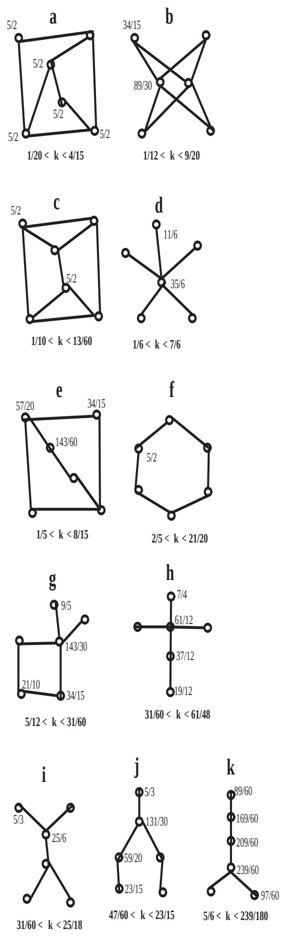


Improving paths (IP)

Jackson & Watts (2002):



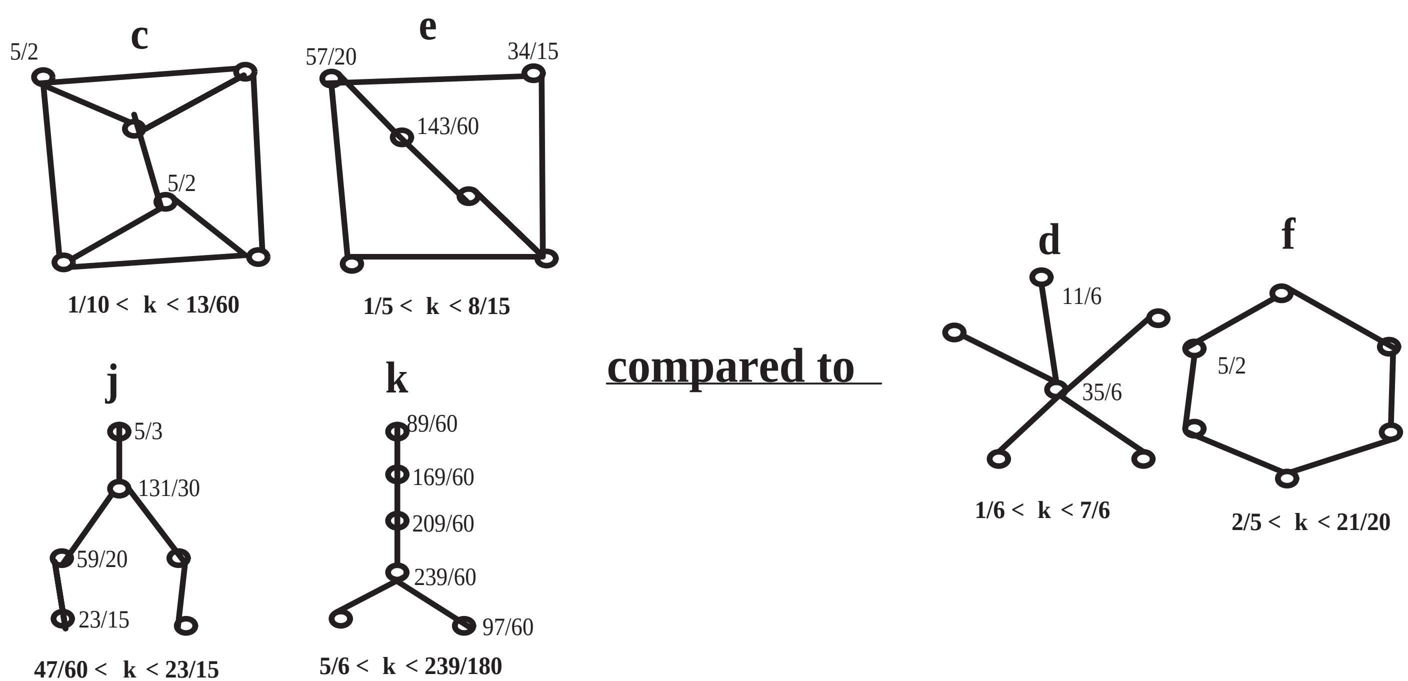
Simulations with IP - results



Starting from...	$k = \frac{1}{5}$	$k = \frac{1}{2}$	$k = 1$
Erdös 5 links	**c** 50^*, **e** 38, **a** 10, **b** 1, **d** 1	**e** 37, **f** 33, **g** 28, **d** 2	**j** 28, **k** 26, **f** 19, **i** 15, **h** 11, **d** 1
Barabási $\mu = 1$ 5 links	**e** 42, **c** 30, **d** 13^{**}, **a** 8, **b** 7	**g** 51^*, **f** 26, **d** 14^{**}, **e** 9^*	**j** 26, **h** 25^*, **d** 11^{**}, **i** 18, **k** 14^*, **f** 6
Erdös 10 links	**c** 63^*, **e** 22^*, **a** 12, **b** 2, **d** 1	**e** 47, **f** 28, **g** 24, **d** 1	**j** 29, **k** 29, **f** 25, **h** 11, **i** 5, **d** 1
Barabási $\mu = 2$ 10 links	**e** 51, **c** 35, **a** 11, **b** 3	**e** 37, **f** 31, **g** 30, **d** 2	**j** 33, **k** 26, **h** 16, **f** 15, **i** 8, **d** 2
Complete 15 links	**c** 72^{**}, **e** 17^{**}, **a** 11	**e** 47, **f** 29, **g** 23, **d** 1	**k** 29, **j** 27, **f** 25, **h** 12, **i** 7

Simulations with IP - results

i) *Unexpected* architectures dominate:



ii) Starting from *preferential attachment*, more <u>efficient</u> networks ($\equiv$ less links) increase (**e** and **c** in column $k = \frac{1}{5}$).

Simulated Annealing (SA)

We test for *stability* as in physics, adding noise.

We start from an equilibrium.
A move is proposed to a, like in IP:

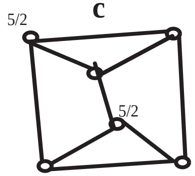
$$\begin{cases} if\ \Delta_a \geq 0 & \text{move is accepted;} \\ if\ \Delta_a < 0 & \text{move is accepted w. p. } \mathbf{e}^{\beta \cdot \mathbf{\Delta_a}}. \end{cases}$$

The process goes on with $\beta \to \infty$ (the *cooling*)
$\Longrightarrow$ we reach another equilibrium.

Difference from original SA: we don't look for global optima.

Simulations with SA - results

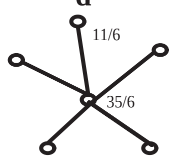
c



1/10 < k < 13/60

(c)	$k=\frac{31}{240}$	$k=\frac{19}{120}$	$k=\frac{3}{16}$
MNE for $\frac{1}{10} > k > \frac{13}{60}$	**c** 66, **a** 20, **b** 8, **l** 8,	**c** 17, a62**, **b** 17, **m** 3	**c** 75, **a** 21, **m** 2, **b** 2

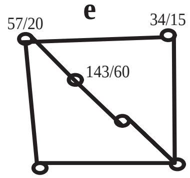
d



1/6 < k < 7/6

Star (d)	$k=\frac{7}{12}$	$k=1$	$k=\frac{17}{12}$
$\frac{1}{6} > k > \frac{11}{6}$	**d** 79, **h** 18, **i** 2, **p** 1	**d** 99, **h** 1	**d** 98, **o** 2

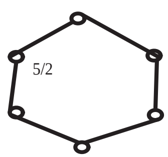
e



1/5 < k < 8/15

(e)	$k=\frac{17}{60}$	$k=\frac{11}{30}$	$k=\frac{27}{60}$
$\frac{1}{5} > k > \frac{8}{15}$	**e** 88, **m** 12	**e** 77, **n** 23	**e** 75, **f** 19, **g** 6

f



2/5 < k < 21/20

Circle (f)	$k=\frac{9}{16}$	$k=\frac{29}{40}$	$k=\frac{71}{80}$
$\frac{2}{5} > k > \frac{21}{20}$	**f** 58, **p** 29, **i** 11, **h** 1	**f** 65, **p** 24, **i** 7, **h** 4	**f** 48, **j** 38, **k** 22, **i** 3

Conclusions

We proposed a model with multiplicity of equilibria.

We used simulations for *qualitative* analysis.

1. Strong *basins of attractions* are not necessarily symmetric nor *stars*;
2. *preferential attachment* (*Barabási*) induces *efficiency*;
3. *asymmetrical* distributions of links induces *stability*.