# Noisy Trading in the Large Market Limit

Mikhail Anufriev
CeNDEF, Department of Economics, University of Amsterdam,
Roetersstraat 11, NL-1018 WB Amsterdam, Netherlands

Giulio Bottazzi
LEM and CAFiM, Sant'Anna School of Advanced Studies,
Piazza Martiri della Libertà 33, 56127 Pisa, Italy

September 16, 2005

# Is it possible to describe the aggregate behavior of a group of heterogeneous agents as if it were a single agent characterized by a "representative" behavior?

Specs:

- pure exchange economy
- dynamics od prevailing prices (and traded quantities)

# 1. Market structure

- Pure exchange two asset economy: one risky (“equity”) and one riskless (“bond”) security.
- trading in discrete time

| Security | Price | Quantity | Unit Payoff | Fixing |
|---|---|---|---|---|
| Bond | 1 | - | $1 + r_f$ | - |
| Equity | $P_t$ | 1 | $P_t\, y_t$ | Walrasian |

Walrasian price fixing: $P_t = \sum_{n=1}^{N} D_{t,n}\,(P_t)$.

| | |
|---|---|
| $r_f$ | riskless interest rate |
| $P_t$ | price of the risky asset |
| $r_{t+1}$ | price return of the risky asset $P_{t+1}/P_t - 1$ |
| $y_t$ | stochastic i.i.d. yield |
| $D_{t,n}$ | demand of risky asset of agent $n$ at time $t$ |

## 2. CRRA framework

Market populated by $N$ heterogenous CRRA-type traders

$$D_{n,t}(P) = x_{n,t}\, W_{n,t}(P) \tag{1}$$

where

| | |
|---|---|
| $W_{n,t}$ | wealth of agent $i$ at time $t$ |
| $x_{n,t}$ | share of wealth invested in the risky asset by agent $n$ at time $t$ |

Motivations of CRRA framework:

- traditional models: Samuelson, Lucas,...
- plausibility:
  - customary in financial advising
  - experimental evidence
  - market impact depends on traders wealth
- completeness: less explored than CARA inside agent based literature (mainly with numerical simulations, but exceptions . . . )

## 2.1. Agent's investment function

Nature of yield process assumed known to all agents.Adaptive investment decisions based on common and public information set $\mathbb{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\}$

**Assumption 1.** For each agent $n$ there exists a parameter $L$ and differentiable *investment function* $f_n$ which maps the last $L$ returns into his investment share:

$$x_{t,n} = f_n(r_{t-1}, r_{t-1}, \dots, r_{t-L}) \quad . \tag{2}$$

N.B.: $x_{i,t}$ independent of present wealth level $W_{i,t}$, past choices $x_{i,t-1}, \dots$, dividend realizations $Y_{t-1}, \dots$.

Generic beliefs and preferences instead of typical adaptive portfolio decision structure

Estimators → Predictions → Investment decisions

# 3. Market dynamics

Wealth at time $t$ for any notional price $P$

$$W_{t,n}(P) = (1 - x_{t-1,n})\, W_{t-1,n}\, (1 + r_f) + \frac{x_{t-1,n}\, W_{t-1,n}}{P_{t-1}}\, (P + y_{t1}\, P_{t-1}) \tag{3}$$

Price return $r_{t+1} = P_{t+1}/P_t - 1$ becomes

$$r_{t+1} = \frac{\left\langle x_{t+1} \right\rangle_t (1 + r_f) - \left\langle x_t \right\rangle_t + (y_t - r_f) \left\langle x_t\, x_{t+1} \right\rangle_t}{\left\langle x_t\, (1 - x_{t+1}) \right\rangle_t} \tag{4}$$

where $\left\langle a \right\rangle_t$ denotes the *wealth weighted average* of variable $a_n$ at time $t$, i.e.

$$\left\langle a \right\rangle_t = \sum_{n=1}^{N} a_n\, \varphi_{t,n}\ , \quad \text{where} \quad \varphi_{t,n} = \frac{W_{t,n}}{W_t} \quad \text{and} \quad W_t = \sum_{n=1}^{N} W_{t,n}. \tag{5}$$

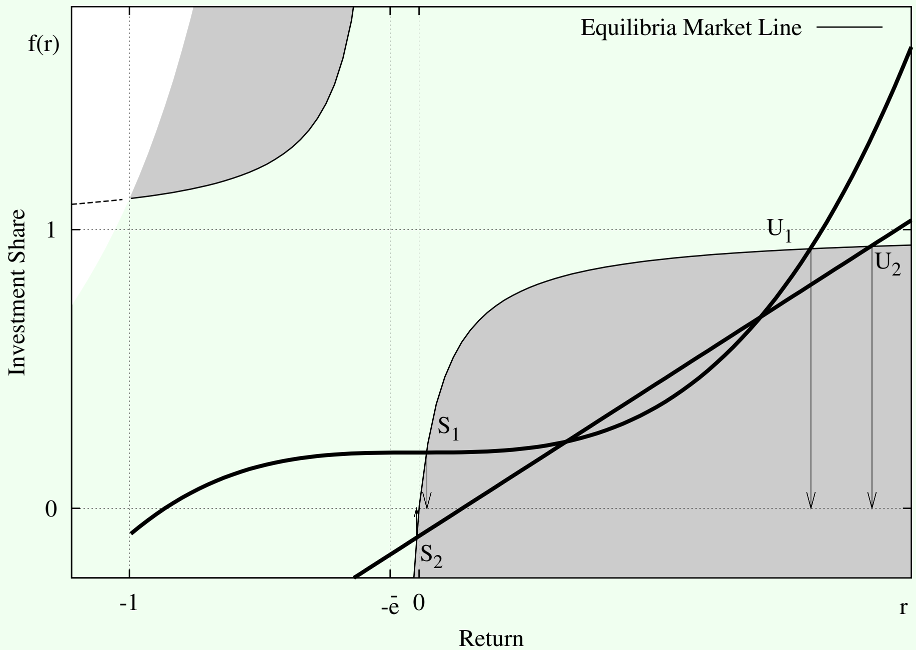


Figure 1: Four equilibria in the market with two agents.

# 4. ex pluribus, unus/a

**Lemma 4.1.** *The wealth-weighted average of market investment choices at time* $t+1$, $\langle x_{t+1} \rangle_{t+1}$ *can be computed using present wealth shares* $\varphi_{t,n}$ *according to*

$$\langle x_{t+1} \rangle_{t+1} = \langle x_{t+1} \rangle_t + (r_{t+1} + e_{t+1}) \left( \langle x_{t+1}\, x_t \rangle_t - \langle x_{t+1} \rangle_t \langle x_t \rangle_t \right) . \quad (6)$$

**Assumption 2.** There exist a function $F$ such that

$$\langle x_t \rangle_t = F(r_{t-1}, r_{t-2}, \dots) \quad . \quad (7)$$

The investment choices at successive time steps satisfy

$$\langle x_t\, x_{t+1} \rangle_t = \langle x_t \rangle_t \langle x_{t+1} \rangle_t \, . \quad (8)$$

and now for a simple example . . .

# 5. The Large Market Limit

The investment decision of each agent is a ”noisy” version of a basic common choice

$$x_{t,n} = F(\mathcal{I}_{t-1}) + \epsilon_{t,n} \ , \tag{9}$$

where the $\epsilon$'s are i.i.d., then

$$\left\langle x_t \right\rangle_t = F(\mathcal{I}_{t-1}) + \left\langle \epsilon_t \right\rangle_t \tag{10}$$

and

$$\left\langle x_t \, x_{t+1} \right\rangle_t = F^2 + F \left( \left\langle \epsilon_t \right\rangle_t + \left\langle \epsilon_{t+1} \right\rangle_t \right) + \left\langle \epsilon_t \epsilon_{t+1} \right\rangle_t \tag{11}$$

The previous assumption is valid only “on average”, i.e.

- when $N \to \infty$
- when the sample average of the trajectories is considered, $< x_t >_t \to \mathrm{E}[< x_t >_t]$

What happens to actual simulations with a finite number of agents? Two sources of noise: $\epsilon$ and $y$ .

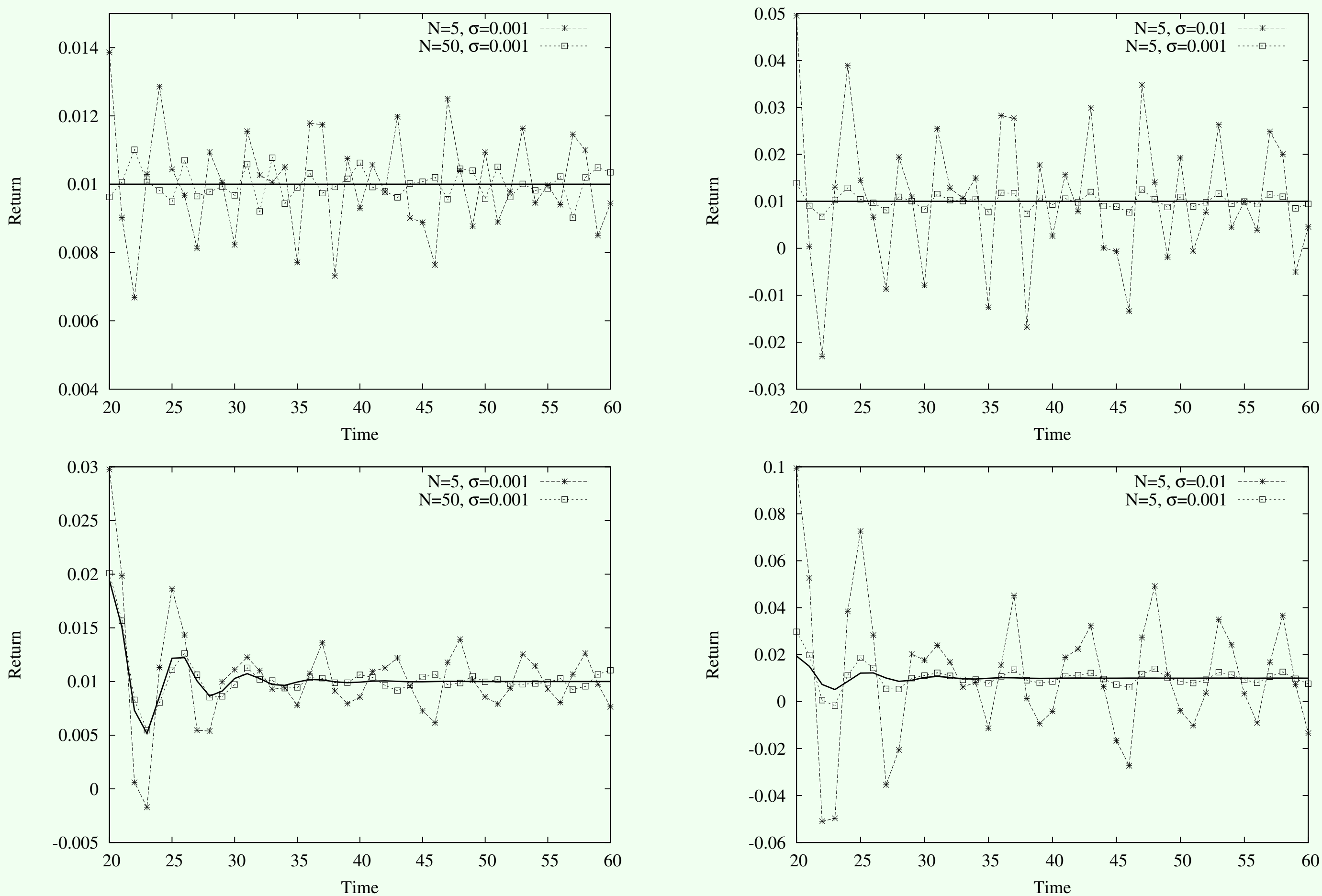


Figure 2: Deterministic dividends. Convergence to LML with the increase of the num. of agents (**Left Panels**) and with the decrease of the variance of the agent-specific noise (**Right Panels**). Simulations for pure noise model (**Upper Panels**) and for model with linear increasing investment function (**Lower Panels**).

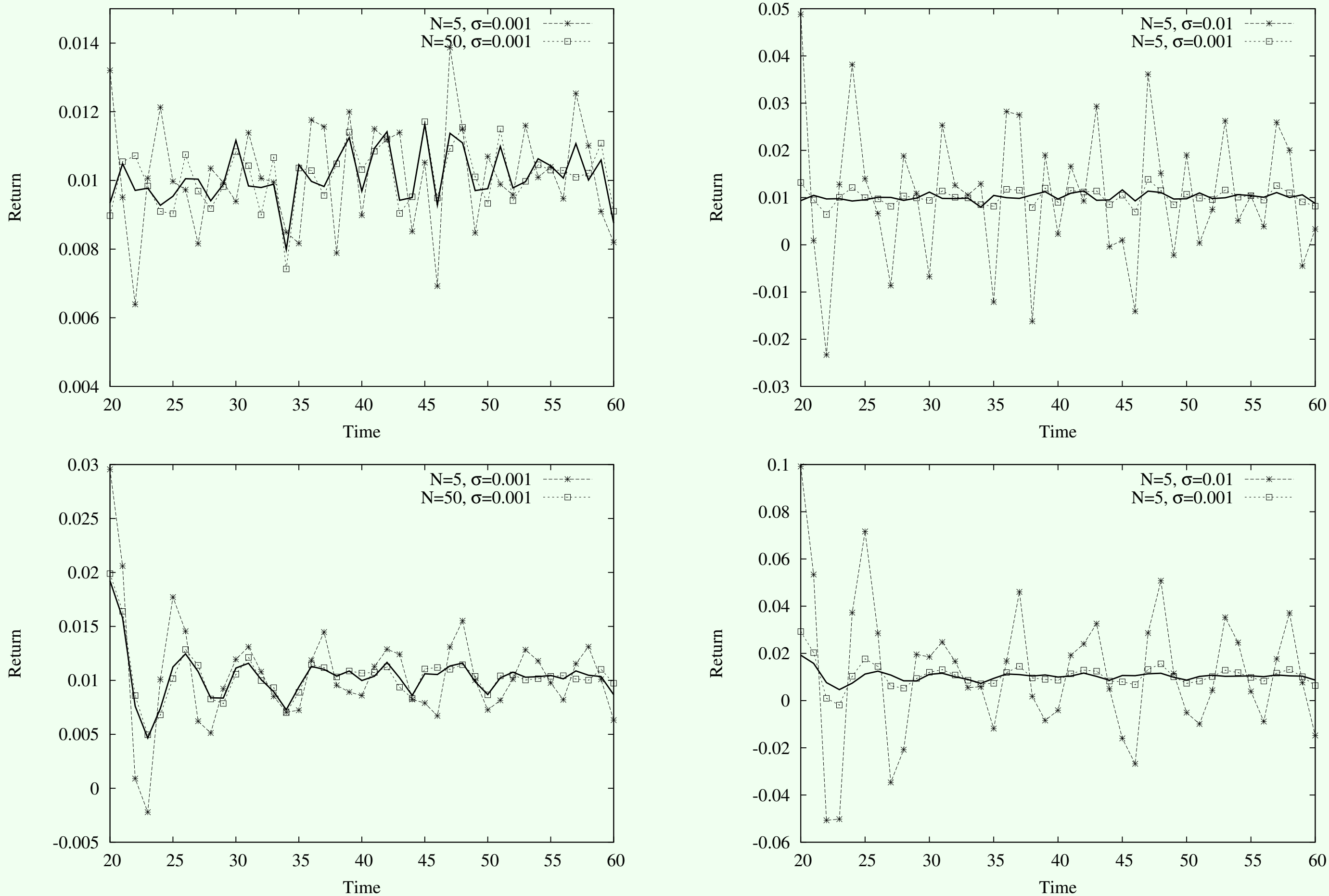


Figure 3: Stochastic dividend. Convergence to the stochastic LML with the increase of the num. of agents (**Left Panels**) and with the decrease of the variance of the agent-specific noise (**Right Panels**). Simulations for pure noise model (**Upper Panels**) and for model with linear increasing strategy (**Lower Panels**).

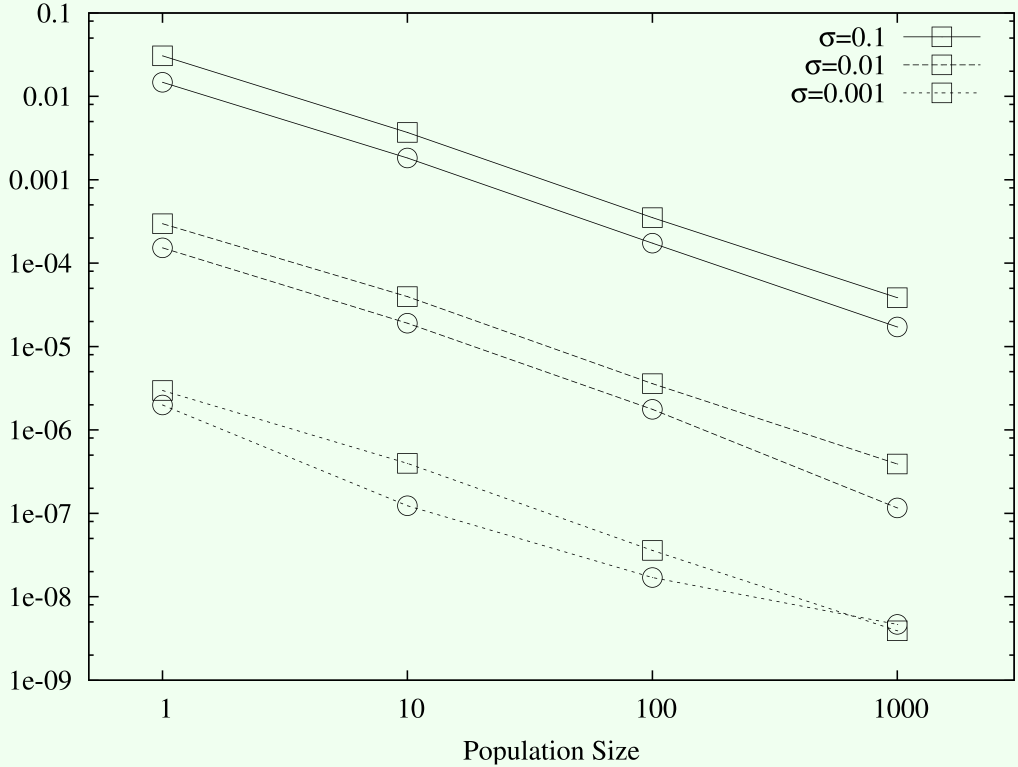


Figure 4: Average over 100 simulations of the sample mean (boxes) and standard deviation (circles) of the deviation of returns from the LML system as function of the number of agents. Different lines correspond to different values of the variance $\sigma_\epsilon^2$.

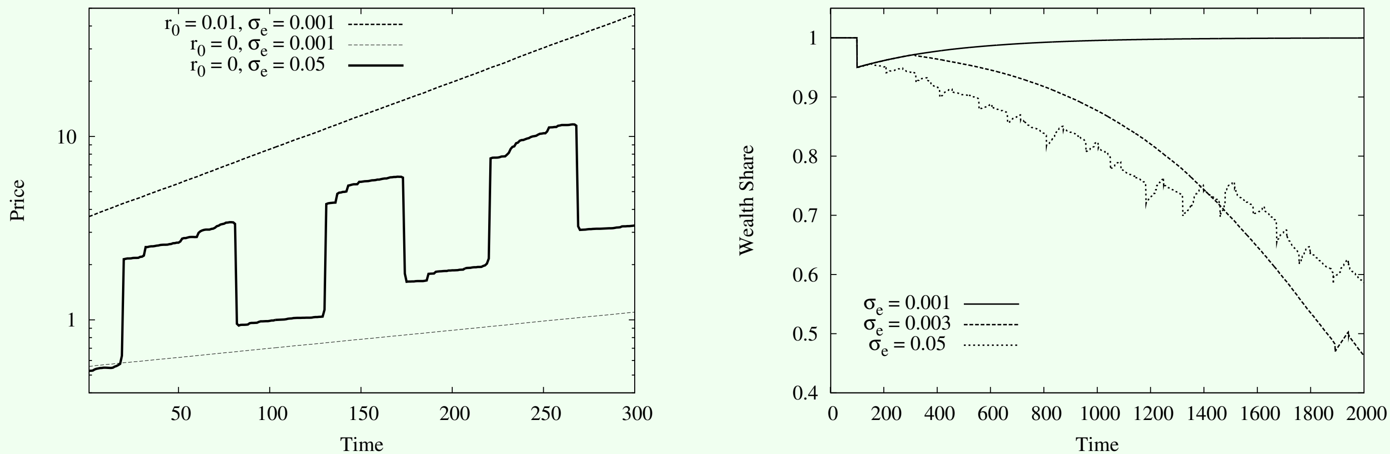


Figure 5: Numerical simulations of the stochastic dynamical system. The yield is log-normally distributed with mean $\bar{e} = 0.02$. **Left panel:** Noise-driven switching between basins of attraction of two different equilibria. Three price trajectories are shown in log-scale for the case of single agent with $S$-shaped investment function and for different initial returns $r_0$ and standard deviations $\sigma_e$. **Right panel:** Dynamics of the relative wealth share $\varphi_{t,1}$ of this trader after the entry of the second trader at time $t = 100$ with initial wealth share $\varphi_{100,2} = 0.05$, for different value of the standard deviations $\sigma_e$